COVER PAGE

Geometry Qualifying Examination

Friday, January 4, 2019
2:30pm – 4:30pm

C-304 Wells Hall

Your Sign-Up Number: ______________________

Note: Attach this cover page to the paperwork you are submitting to be graded. **This number should be the only identification appearing on all of your paperwork – DO NOT WRITE YOUR NAME on any of the paperwork you are submitting.**

Geometry Qualifying Exam

Michigan State University – Fall 2018

Instructions. Solve any 4 of the following 5 problems. Even if you attempt more than 4 problems, indicate which problems you want graded. You must justify your claims either by direct arguments or by referring to specific theorems that you know.

All manifolds, functions, vector fields, etc. are assumed to be smooth.

Problem 1. Let S be the zero locus of the cubic polynomial $p(z,w) = z(z^2+w)+w$ in $\mathbb{C}^2$:

$$S \;=\; \Big\{(z,w)\in\mathbb{C}^2 \;\Big|\; z(x^2+w)+w=0\Big\}.$$

Identify $\mathbb{C}^2$ with $\mathbb{R}^4$ by $(z,w)=(s+it, u+iv)$. Prove that S is an embedded submanifold of $\mathbb{R}^4$.

Problem 2. Let M be a manifold of dimension n.

(a) Complete the definition: A *vector bundle* of rank k over M is a manifold E together with a map ...

(b) Complete the definition: A *section* of E is ...

(c) Prove that the tangent bundle TM is a vector bundle over M.

Problem 3. Let M be an n-dimensional manifold. We say that $S\subset M$ is a k-dimensional embedded submanifold if, intuitively, "S is locally equivalent to the canonical embedding $\mathbb{R}^k \hookrightarrow \mathbb{R}^n$".

(a) Give a complete, precise definition: A subset $S\subset M$ is a *k-dimensional embedded submanifold* if ...

(b) Prove that if S is compact and X is a vector field on S, then there exists a smooth vector field $\widetilde{X}$ on M such that the restriction of $\widetilde{X}$ to S is X.

(c) Give an example which shows that (b) is not true in general if S is not compact.

Problem 4. Consider $M=\mathbb{R}^4\setminus\{0\}$ with the standard euclidean metric and with coordinates (x,y,z,w). As usual, orient M by $dx\wedge dy\wedge dz\wedge dw$, and define r by $r^2=x^2+y^2+z^2+w^2$.

(a) Write the 1-form $\alpha = d\Big(\frac{1}{r^2}\Big)$ in terms of r, dx, dy, dz, and dw (but not dr).

(b) Using (a), compute $\beta = *\,\alpha$, where $*$ is the Hodge star operator. Write your answer in the form

$$\beta \;=\; \frac{c}{r^k}\,\gamma \tag{1}$$

for some constants c and k and some 3-form γ that is smooth across the origin.

(c) Calculate $\int_{S^3} \beta$, where $S^3 \subset \mathbb{R}^4$ is the unit sphere, oriented as the boundary of the unit ball B^4.

Hint: Use (1) and the note that $Vol\, S^3 = 2\pi^2$ *and* $Vol\, B^4 = \frac{1}{2}\pi^2$.

(d) Show that $d\beta = 0$.

(e) What is $\int_{\partial K} \beta$, where K is the unit cube $K = [-1, 1]^4 \subset \mathbb{R}^4$? *Hint: Sketch* S^3 *and* ∂K.

Problem 5. Let M be a manifold.

(a) Complete the definition: A *Riemannian metric* on M is a type $(0, 2)$ tensor field g such that ...

(b) Let g_0 and g_1 be two Riemannian metrics on M. Prove that $g_t = (1-t)g_0 + tg_1$ is a Riemannian metric for all $t \in [0, 1]$.

(c) Complete the definition: A topological space X is *contractible* if there exists a map ...

(d) Use (b) to explain why the space $\mathcal{R}$ of all Riemannian metrics on a manifold M is contractible. *You do not need to prove that your maps are continuous.*